

# Metrically Ramsey ultrafilters

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## Abstract

Given a metric space  $(X, d)$ , we say that a mapping  $\chi : [X]^2 \rightarrow \{0, 1\}$  is an isometric coloring if  $d(x, y) = d(z, t)$  implies  $\chi(\{x, y\}) = \chi(\{z, t\})$ . A free ultrafilter  $\mathcal{U}$  on an infinite metric space  $(X, d)$  is called metrically Ramsey if, for every isometric coloring  $\chi$  of  $[X]^2$ , there is a member  $U \in \mathcal{U}$  such that the set  $[U]^2$  is  $\chi$ -monochrome. We prove that each infinite ultrametric space  $(X, d)$  has a countable subset  $Y$  such that each free ultrafilter  $\mathcal{U}$  on  $X$  satisfying  $Y \in \mathcal{U}$  is metrically Ramsey. On the other hand, it is an open question whether every metrically Ramsey ultrafilter on the natural numbers  $\mathbb{N}$  with the metric  $|x - y|$  is a Ramsey ultrafilter. We prove that every metrically Ramsey ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  has a member with no arithmetic progression of length 2, and if  $\mathcal{U}$  has a thin member then there is a mapping  $f : \mathbb{N} \rightarrow \omega$  such that  $f(\mathcal{U})$  is a Ramsey ultrafilter.

**Classification:** 03E05, 54E35

**Keywords:** selective ultrafilter, metrically Ramsey ultrafilter, ultrametric space.

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Let  $X$  be an infinite set and let  $\mathfrak{F}$  be some family of  $\{0, 1\}$ -colorings of the set  $[X]^2$  of all two-element subsets of  $X$ . We say that a free ultrafilter  $\mathcal{U}$  on  $X$  is *Ramsey with respect to  $\mathfrak{F}$*  if, for any coloring  $\chi \in \mathfrak{F}$ , there exists  $U \in \mathcal{U}$  such that  $[U]^2$  is  $\chi$ -monochrome. In the case in which  $\mathfrak{F}$  is the family of all  $\{0, 1\}$ -colorings of  $[X]^2$ , we get the classical definition of *Ramsey ultrafilters*. It is well-known that  $\mathcal{U}$  is a Ramsey ultrafilter if and only if  $\mathcal{U}$  is *selective*, i.e. for

every partition  $\mathcal{P}$  of  $X$  either  $P \in \mathcal{U}$  for some  $P \in \mathcal{P}$  or there exists  $U \in \mathcal{U}$  such that  $|U \cap P| \leq 1$  for each  $P \in \mathcal{P}$ .

Given a metric space  $(X, d)$ , we say that a mapping  $\chi : [X]^2 \rightarrow \{0, 1\}$  is an *isometric coloring* if  $d(x, y) = d(z, t)$  implies  $\chi(\{x, y\}) = \chi(\{z, t\})$ . We note that every isometric coloring  $\chi$  is uniquely defined by some mapping  $f : d(X, X) \setminus \{0\} \rightarrow \{0, 1\}$ . Indeed, we take an arbitrary  $r \in d(X, X) \setminus \{0\}$ , choose  $\{x, y\} \in [X]^2$  such that  $d(x, y) = r$  and put  $f(r) = \chi(\{x, y\})$ . On the other hand, for  $f : d(X, X) \setminus \{0\} \rightarrow \{0, 1\}$ , we define  $\chi$  by  $\chi(\{x, y\}) = f(d(x, y))$ .

We say that a free ultrafilter on an infinite metric space  $(X, d)$  is *metrically Ramsey* if  $\mathcal{U}$  is Ramsey with respect to all isometric colorings of  $[X]^2$ .

Let  $G$  be a group and let  $X$  be a  $G$ -space with the action  $(G, X) \rightarrow X, (g, x) \mapsto gx$ . A coloring  $\chi : [X]^2 \rightarrow \{0, 1\}$  is called  *$G$ -invariant* if  $\chi(\{x, y\}) = \chi(\{gx, gy\})$  for all  $\{x, y\} \in [X]^2$  and  $g \in G$ . A free ultrafilter  $\mathcal{U}$  of  $X$  is called  *$G$ -Ramsey* if  $\mathcal{U}$  is Ramsey with respect to the family of all  $G$ -invariant colorings of  $[X]^2$ .

We consider the special case:  $X$  is a metric space and  $G$  is a group of isometries of  $X$ . Clearly, every isometric coloring of  $[X]^2$  is  $G$ -invariant. If  $G$  is metrically 2-transitive (if  $d(x, y) = d(z, t)$  then there is  $g \in G$  such that  $g\{x, y\} = \{z, t\}$ ) then every  $G$ -invariant coloring of  $[X]^2$  is an isometric coloring.

We take the group  $\mathbb{Z}$  of integers, put  $X = \mathbb{Z}$  and consider the action  $\mathbb{Z}$  on  $X$  by  $(g, x) = g + x$ . *Is every  $\mathbb{Z}$ -Ramsey ultrafilter selective?* This question appeared in [5] and, to our knowledge, remains open. We endow  $\mathbb{Z}$  with the metric  $d(x, y) = |x - y|$ . By

above paragraph an ultrafilter  $\mathcal{U}$  on  $\mathbb{Z}$  is  $\mathbb{Z}$ -Ramsey if and only if  $\mathcal{U}$  is metrically Ramsey. *Is every metrically Ramsey ultrafilter on  $\mathbb{Z}$ -selective?* This is an equivalent form of the above question. The case of  $\mathbb{Z}$  evidently equivalent to the case of  $\mathbb{N}$ .

Surprisingly or not, the case of ultrametric spaces is cardinally different and much more easy to explore. We recall that a metric  $d$  is an *ultrametric* if  $d(x, y) \leq \max\{d(x, z), d(z, y)\}$  for all  $x, y, z \in X$ . We prove that every infinite ultrametric space  $X$  has a countable subset  $Y$  such that any ultrafilter  $\mathcal{U}$  on  $X$  satisfying  $Y \in \mathcal{U}$  is metrically Ramsey.

## 1. Equidistance subsets

We say that a subset  $Y$  of a metric space  $(X, d)$  is an *equidistance subset* if there is  $r \in \mathbb{R}^+$  such that  $d(x, y) = r$  for all distinct  $x, y \in Y$ . If  $Y$  is an equidistance subset of  $(X, d)$  then every free ultrafilter  $\mathcal{U}$  on  $X$  such that  $Y \in \mathcal{U}$  is metrically Ramsey.

**Proposition 1.1.** *Every infinite metric space with finite scale  $d(X, X)$ ,  $d(X, X) = \{d(x, y) : x, y \in X\}$  has a countable equidistance subset.*

*Proof.* We define a coloring  $\chi : [X]^2 \longrightarrow d(X, X)$  by  $\chi(\{x, y\}) = d(x, y)$  and apply the classical Ramsey theorem [2, p.16].  $\square$

For an ultrametric space  $(X, d)$  and  $r \in d(X, X)$ , we use the equivalence  $\sim_r$  defined by  $x \sim_r y$  if and only if  $d(x, y) \leq r$ . Then  $X$  is partitioned into classes of  $r$ -equivalence  $X = \bigcup_{\alpha < \lambda} X_\alpha$ . If

$x, y \in X_\alpha$  then  $d(x, y) \leq r$ . If  $x \in X_\alpha$ ,  $y \in X_\beta$  and  $\alpha \neq \beta$  then  $d(x, y) > r$ .

**Proposition 1.2.** *Let  $(X, \alpha)$  be an infinite ultrametric space with finite scale  $d(X, X)$ . If  $|X|$  is regular then  $X$  has an equidistance subset  $Y$  of cardinality  $|Y| = |X|$ . If  $|X|$  is singular then, for every cardinal  $\kappa < |X|$  there is an equidistance subset of cardinality  $\kappa$ .*

*Proof.* Let  $d(X, X) = \{0, r_1, \dots, r_n\}$ ,  $0 < r_1 < \dots < r_n$ . We proceed on induction by  $n$ . For  $n = 1$ , the statement is evident:  $Y = X$ .

To make the inductive step from  $n$  to  $n + 1$ , we partition  $X$  into classes of  $r_n$ -equivalence  $X = \bigcup_{\alpha < \lambda} X_\alpha$ . If  $\lambda = |X|$  then we pick one element  $y_\alpha \in X_\alpha$  and put  $Y = \{y_\alpha : \alpha < |X|\}$ , so  $d(x, y) = r_{n+1}$  for all distinct  $x, y \in Y$ . Assume that  $\lambda < |X|$ . If  $|X|$  is regular, we take  $\alpha$  so that  $|X_\alpha| = X$  and apply the inductive assumption to  $X_\alpha$ . If  $|X|$  is singular then we take  $X_\alpha$  such that  $|X_\alpha| > \kappa$  and use the inductive assumption.  $\square$

**Remark 1.1.** If in Proposition 1.2.  $|X|$  is singular, we cannot state that there is an equidistance subset  $Y$  of cardinality  $X$ . We take an arbitrary singular cardinal  $\kappa$ , put  $X = \kappa$  and partition  $X = \bigcup_{\alpha < \lambda} X_\alpha$  so that  $\lambda < \kappa$  and  $|X_\alpha| < \kappa$  for each  $\alpha < \kappa$ . We define an ultrametric  $d$  on  $X$  by  $d(x, x) = 0$ ,  $d(x, y) = 1$  if  $x, y \in X_\alpha$ ,  $x \neq y$  and  $d(x, y) = 2$  if  $x \in X_\alpha$ ,  $y \in X_\beta$ ,  $\alpha \neq \beta$ . If  $Y$  is an equidistance subset of  $(X, \alpha)$  then either  $Y \subseteq X_\alpha$  for some  $\alpha < \lambda$ , or  $|Y \cap X_\alpha| \leq 1$  for each  $\alpha < \lambda$ . Hence,  $|Y| < \kappa$ .

**Proposition 1.3.** *For every infinite cardinal  $\kappa$ , there exists a metric space  $(X, d)$  such that  $|X| = 2^\kappa$ ,  $d(X, X) = \{0, 1, 2\}$  and every equidistance subset  $Y$  of  $(X, d)$  is of cardinality  $|Y| \leq \kappa$ .*

*Proof.* We put  $X = 2$  and apply [4, Theorem 6.2] to define a coloring  $\chi : [X]^2 \longrightarrow \{1, 2\}$  with no monochrome  $[Z]^2$  for  $|Z| > \kappa$ . Then we define a metric  $d$  on  $X$  by  $d(x, x) = 0$  and  $d(x, y) = \chi(x, y)$  for all distinct  $x, y \in X$ .  $\square$

**Proposition 1.4.** *Let  $(X, d)$  be a metric space with infinite scale  $d(X, X)$ ,  $|d(X, X)| = \kappa$ . If  $|X| \geq (2^\kappa)^+$  then there is an equidistance subset  $Y$  of  $(X, d)$  such that  $|Y| = \kappa^+$ .*

*Proof.* We define a coloring  $\chi : [X]^2 \longrightarrow d(X, X)$  by  $\chi(\{x, y\}) = d(x, y)$  and apply the Erdős-Rado theorem [4, Theorem 6.4].  $\square$

**Proposition 1.5.** *For every infinite metric space  $(X, d)$ , there exists an injective sequence  $(x_n)_{n \in \omega}$  in  $X$  such that one of the following conditions is satisfied:*

- (i) *the sequence  $(d(x_0, x_n))_{n \in \omega}$  is increasing;*
- (ii) *the sequence  $(d(x_0, x_n))_{n \in \omega}$  is decreasing;*
- (iii) *for every  $n \in \omega$  and all  $i, j$ ,  $i > n$ ,  $j > n$ , we have  $d(x_n, x_i) = d(x_n, x_j)$ .*

*Proof.* We assume that there exists  $x_0 \in X$  such that the set  $d(x_0, X)$  is infinite,  $d(x_0, X) = \{d(x_0, x) : x \in X\}$ . We choose a countable subset  $Y$  of  $X$  such that  $x_0 \notin Y$  and  $d(x_0, y) \neq d(x_0, z)$  for all distinct  $y, z \in Y$ . The set  $d(x_0, Y)$  contains either increasing or decreasing sequence  $(r_{n+1})_{n \in \omega}$ . For each  $n \in \omega$ , we choose  $x_{n+1}$  such

that  $d(x_0, x_{n+1}) = r_n$ . Then the sequence  $(d(x_0, x_n))_{n \in \omega}$  satisfies either (i) or (ii).

In the alternative case, the set  $d(x, X)$  is finite for each  $x \in X$ . We fix  $x_0 \in X$  and choose a countable subset  $X_1$  such that  $|d(x_0, X_1)| = 1$ . We pick  $x_1 \in X_1$  and choose a countable subset  $X_2 \subseteq X_1$  such that  $|d(x_1, X_2)| = 1$  and so on. After  $\omega$  steps, we get the sequence  $(x_n)_{n \in \omega}$  satisfying (iii).  $\square$

**Proposition 1.6.** *Let  $(X, d)$  be an infinite metric space and let  $\{x_n : n \in \omega\}$  be a family of non-empty pairwise disjoint subsets of  $X$  such that  $d(X_i, X_j) \cap d(X_n, X_n) = \emptyset$  for all  $n$  and distinct  $i, j$ . Let  $\mathcal{U}$  be a metrically Ramsey ultrafilter on  $X$  such that  $\bigcup_{n < \omega} X_n \in \mathcal{U}$  and  $X_n \notin \mathcal{U}$  for each  $n < \omega$ . Then the following statements hold:*

- (i) *there exists  $U \in \mathcal{U}$  such that  $|U \cap X_n| \leq 1$  for each  $n < \omega$ ;*
- (ii) *if  $d(X_i, X_j) \cap d(X_k, X_l) = \emptyset$  for all distinct  $\{i, j\}, \{k, l\} \in [\omega]^2$  then there is a mapping  $\varphi : X \rightarrow \omega$  such that the ultrafilter  $\varphi(\mathcal{U})$  is selective;*
- (iii) *if for each  $n < \omega$  there exists  $m < \omega$  such that  $m > n$ ,  $|X_m| > n$ , then there exists an ultrafilter  $\mathcal{V}$  on  $X$  such that  $\bigcup_{n < \omega} X_n \in \mathcal{V}$  and  $\mathcal{V}$  is not metrically Ramsey.*

*Proof.* (i) By the assumption, the sets  $A = \bigcup_{n < \omega} d(X_n, X_n)$  and  $B = \bigcup_{i \neq j} d(X_i, X_j)$  are disjoint. We take an arbitrary mapping  $f : \mathbb{R}^+ \rightarrow \{0, 1\}$  such that  $f|_A \equiv 0$ ,  $f|_B \equiv 1$ , and consider the isometric coloring  $\chi$  of  $[X]^2$  defined by  $f$ . Since  $\mathcal{U}$  is metrically Ramsey, there is  $U \in \mathcal{U}$  such that  $[U]^2$  is  $\chi$ -monochrome. Clearly,

$|U \cap X_n| \leq 1$  for each  $n < \omega$ .

(ii) We define  $\varphi$  by the rule: if  $x \in X_i$  then  $\varphi(x) = i$ , if  $x \in X \setminus \bigcup_{n < \omega} X_n$  then  $\varphi(x) = 0$ . We take an arbitrary coloring  $\chi' : [\omega]^2 \longrightarrow \{0, 1\}$  and define a coloring  $\chi : [\bigcup_{n < \omega} X_n]^2 \longrightarrow \{0, 1\}$  as follows. If  $x \in X_i, y \in X_j, i \neq j$  then  $\chi(\{x, y\}) = \chi'(\{i, j\})$ . If  $x, y \in X_n$  then  $\chi(\{x, y\}) = 0$ . By the assumption, the coloring  $\chi$  is isometric. We choose  $U \in \mathcal{U}$  such that  $U \subseteq \bigcup_{n < \omega} X_n$  and  $[U]^2$  is  $\chi$ -monochrome and  $|U \cap X_n| \leq 1$  for each  $n < \omega$ . Then  $\varphi(U) \in \varphi(\mathcal{U})$  and  $[\varphi(U)]$  is  $\chi'$ -monochrome, so  $\varphi(\mathcal{U})$  is a Ramsey ultrafilter.

(iii) We consider the family of all filters  $\mathfrak{F}$  on  $X$  such that  $\bigcup_{n < \omega} X_n \in \mathfrak{F}$  and, for every  $n \in \omega$  and  $F \in \mathfrak{F}$ , there exists  $m \in \omega$  such that  $|F \cap X_m| > n$ . By the Zorn Lemma, this family has maximal by inclusion element  $\mathcal{V}$ . It is easy to verify that  $\mathcal{V}$  is ultrafilter. By (i),  $\mathcal{V}$  is not metrically Ramsey.  $\square$

## 2. The ultrametric case

**Proposition 2.1.** *For every infinite ultrametric space  $(X, d)$ , there exists a countable subset  $Y$  of  $X$  such that every free ultrafilter  $\mathcal{U}$  on  $X$  satisfying  $Y \in \mathcal{U}$  is metrically Ramsey.*

*Proof.* We choose the sequence  $(x_n)_{n \in \omega}$  given by Proposition 1.5, put  $Y = \{x_n : n \in \omega\}$ , fix an arbitrary mapping  $f : \mathbb{R}^+ \longrightarrow \{0, 1\}$  and take an arbitrary free ultrafilter  $\mathcal{U}$  satisfying  $Y \in \mathcal{U}$ .

We assume that either (i) or (ii) of Proposition 1.5 hold for  $(x_n)_{n \in \omega}$ . We define a mapping  $h : Y \longrightarrow \mathbb{R}^+$  by  $h(x_n) = d(x_0, x_n)$

and choose  $k \in \{0, 1\}$  such that  $(fh)^{-1}(k) \in \mathcal{U}$ . Since  $d$  is an ultrametric, in the case (i) we have  $d(x_i, x_n) = d(x_0, x_n)$  for all  $i < n$ , and in the case (ii) we have  $d(x_i, x_n) = d(x_0, x_i)$  for all  $i < n$ . In both cases, if  $\{x_i, x_n\} \in [(fh)^{-1}(k)]^2$  then  $f(d(x_i, x_n)) = k$ .

If  $(x_n)_{n \in \omega}$  satisfies (iii) of Proposition 1.5 then we define a mapping  $h : Y \rightarrow \mathbb{R}^+$  by  $h(x_n) = d(x_n, x_i)$ ,  $i > n$  and repeat above arguments.  $\square$

**Proposition 2.2.** *For a free ultrafilter  $\mathcal{U}$  on an infinite set  $X$ , the following statements are equivalent:*

- (i)  $\mathcal{U}$  is selective;
- (ii)  $\mathcal{U}$  is metrically Ramsey for each ultrametric  $d$  on  $X$  such that  $d(X, X) = \{0, 1, 2\}$ .

*Proof.* The implication (i)  $\implies$  (ii) is evident. We assume that  $\mathcal{U}$  is not selective and choose a partition  $\mathcal{P}$  of  $X$  such that  $P \notin \mathcal{U}$  for each  $P \in \mathcal{P}$ , and for every  $U \in \mathcal{U}$ , there is  $P \in \mathcal{P}$  such that  $|P \cap U| > 1$ . We define an ultrametric  $d$  on  $X$  by  $d(x, x) = 0$ ,  $d(x, y) = 1$  if  $x \neq y$ ,  $x, y \in P$  for some  $P \in \mathcal{P}$ , and  $d(x, y) = 2$  if  $x, y$  belong to different cells of the partition  $\mathcal{P}$ . We define a coloring  $\chi : [X]^2 \rightarrow \{1, 2\}$  by  $\chi(\{x, y\}) = d(x, y)$ . Then the set  $[U]^2$  is not  $\chi$ -monochrome for each  $U \in \mathcal{U}$  so  $\mathcal{U}$  is not metrically Ramsey and (ii)  $\implies$  (i).  $\square$

**Proposition 2.3.** *Let  $(X, d)$  be an infinite ultrametric space with finite scale  $d(X, X) = \{0, r_1, \dots, r_n\}$ ,  $0 < r_1 < \dots < r_n$ . Then the following statements are equivalent:*



(i) every free ultrafilter on  $(X, d)$  is metrically Ramsey;

(ii) for every  $i \in \{1, \dots, n\}$ , the partition  $\mathcal{P}_i$  of  $X$  into classes of  $r_i$ -equivalence has only finite number of infinite classes and there is  $m \in \omega$  such that  $|C| < m$  for each finite class  $C$  from  $\mathcal{P}_i$ .

*Proof.* (i)  $\implies$  (ii). If  $\mathcal{P}_i$  has infinitely many infinite classes or the set  $\{|C| : C \text{ is a finite class from } \mathcal{P}_i\}$  is infinite we apply Proposition 1.6(iii) to get a free ultrafilter  $\mathcal{V}$  on  $X$  which is not metrically Ramsey.

(ii)  $\implies$  (i). We proceed on induction by  $n$ . For  $n = 1$ , the statement is evident.

To make the inductive step from  $n$  to  $n + 1$ , take an arbitrary free ultrafilter  $\mathcal{U}$  on  $X$  and we consider the partition  $\mathcal{P}_{n+1}$ . Let  $X_1, \dots, X_m$  be the set of all infinite classes from  $\mathcal{P}_{n+1}$ . If  $X_1 \cup \dots \cup X_m \in \mathcal{U}$  then we take  $X_i \in \mathcal{U}$  and apply the inductive assumption. If  $X \setminus (X_1 \cup \dots \cup X_m) \in \mathcal{U}$  then we choose  $U \in \mathcal{U}$  such that  $U \subseteq X \setminus (X_1 \cup \dots \cup X_m)$  and  $|U \cap C| \leq 1$  for each finite class  $C \in \mathcal{P}_{n+1}$ . Then  $U$  is an equidistance set so  $\mathcal{U}$  is metrically Ramsey.  $\square$

### 3. The case of $\mathbb{N}$

**Proposition 3.1.** *Let  $\mathcal{U}$  be a metrically Ramsey ultrafilter on  $\mathbb{N}$  and let  $f : \mathbb{N} \longrightarrow \mathbb{N}$  be a mapping such that  $f(x) > x$  for each  $x \in \mathbb{N}$ . Then there exists a member  $U \in \mathcal{U}$  having no subsets of the form  $\{a, a + x, a + f(x)\}$ . In particular ( $f(x) = 2x$ ), some member of  $\mathcal{U}$  has no arithmetic progressions of length 2.*

*Proof.* We consider a directed graph  $\Gamma_f$  with the set of vertices  $\mathbb{N}$  and the set of edges  $\{(x, f(x)) : x \in \mathbb{N}\}$ . Since  $f(x) > x$ ,  $\Gamma_f$  is the disjoint union of directed trees  $T$  such that each vertex of  $T$  has at most one input edges. Using this observation, it is easy to partition  $\mathbb{N} = A_1 \cup A_2$  so that  $f(A_1) \subseteq A_2$ ,  $f(A_2) \subseteq A_1$ .

The partition  $\mathbb{N} = A_1 \cup A_2$  defines an isometric coloring  $\chi : [\mathbb{N}]^2 \rightarrow \{1, 2\}$  by  $\chi(\{x, y\}) = i$  if and only if  $d(x, y) \in A_i$ . We take a subset  $U \in \mathcal{U}$  such that the set  $[U]^2$  is  $\chi$ -monochrome and assume that  $\{a, a+x, a+f(x)\} \subset U$  for some  $a, x \in \mathbb{N}$ . We note that  $d(a, a+x) = x$ ,  $d(a, a+f(x)) = f(x)$ , but  $x$  and  $f(x)$  belong to different subsets  $A_1, A_2$ , so  $\chi(\{a, a+x\}) \neq \chi(\{a, a+f(x)\})$  and we get a contradiction with the choice of  $U$ .  $\square$

Let  $\mathcal{U}$  be metrically Ramsey ultrafilter on  $\mathbb{N}$ . Assume that there is  $U \in \mathcal{U}$  such that  $d(x, y) \neq d(z, t)$  for all distinct  $\{x, y\}, \{z, t\} \in [U]^2$ . Then every  $\{0, 1\}$ -coloring of  $[U]^2$  can be extended to some isometric coloring of  $[\mathbb{N}]^2$ . Hence,  $\mathcal{U}$  is a Ramsey ultrafilter.

We say that a subset  $T = \{t_n : t_n < t_{n+1}, n < \omega\}$  of  $\mathbb{N}$  is *thin* if  $(t_{n+1} - t_n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Proposition 3.2.** *If a metrically Ramsey ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  has a thin subset  $T \in \mathcal{U}$  then there exists a mapping  $\varphi : \mathbb{N} \rightarrow \omega$  such that the ultrafilter  $\varphi(\mathcal{U})$  is selective and  $\varphi$  is finite-to-one on some member  $U \in \mathcal{U}$ .*

*Proof.* Let  $T = \{t_n : t_n < t_{n+1}, n \in \omega\}$ . Assume that we have chosen two sequences  $(a_n)_{n \in \omega}$ ,  $(b_n)_{n \in \omega}$  in  $T$  such that

- (1)  $a_n < b_n < a_{n+1} < b_{n+1}$  for each  $n \in \omega$ ;

- (2)  $d([a_n, b_n) \cap T, [a_n, b_n) \cap T) \cap d([a_i, b_i) \cap T, [a_j, b_j) \cap T) = \emptyset$  for all  $n$  and distinct  $i, j$  ;
- (3)  $d([a_i, b_i) \cap T, [a_j, b_j) \cap T) \cap d([a_k, b_k) \cap T, [a_l, b_l) \cap T) = \emptyset$  for all distinct  $\{i, j\}, \{k, l\} \in [\omega]^2$  ;
- (4)  $d([b_n, a_{n+1}) \cap T, [b_n, a_{n+1}) \cap T) \cap d([b_i, a_{i+1}) \cap T, [b_j, a_{j+1}) \cap T) = \emptyset$  for all  $n$  and distinct  $i, j$  ;
- (5)  $d([b_i, a_{i+1}) \cap T, [b_j, a_{j+1}) \cap T) \cap d([b_k, a_{k+1}) \cap T, [b_l, a_{l+1}) \cap T) = \emptyset$  for all distinct  $\{i, j\}, \{k, l\} \in [\omega]^2$  ;

We put  $A = \bigcup_{n \in \omega} ([a_n, b_n) \cap T)$ ,  $B = \bigcup_{n \in \omega} ([b_n, a_{n+1}) \cap T)$  and note that  $A$  and  $B$  with corresponding partitions satisfy Proposition 1.6 (ii). Since either  $A \in \mathcal{U}$  or  $B \in \mathcal{U}$ , Proposition 1.6 (ii) gives the mapping  $\varphi : \mathbb{N} \rightarrow \omega$  such that  $\varphi(\mathcal{U})$  is selective. By the construction of  $\varphi$ ,  $\varphi$  is finite-to-one on  $A$  or  $B$  respectively.

It remains to construct  $(a_n)_{n \in \varphi}$  and  $(b_n)_{n \in \varphi}$ . We put  $a_0 = t_0$ ,  $b_0 = t_1$  and assume that we have chosen  $a_0, b_0, \dots, a_n, b_n$ . Since  $T$  is thin, we can choose  $a_{n+1} \in T$  so that  $a_{n+1} > 2b_n$  and  $|t - t'| > 2a_n$  for all distinct  $t, t' \in T \setminus [1, a_{n+1})$ . Then we choose  $b_{n+1} \in T$  so that  $b_{n+1} > 2a_{n+1}$  and  $|t - t'| > 2b_n$  for all distinct  $t, t' \in T \setminus [1, b_{n+1})$ . After  $\omega$  steps, we get the desired  $(a_n)_{n \in \varphi}, (b_n)_{n \in \varphi}$ .  $\square$

#### 4. Comments and open questions

1. In connection with Proposition 3.1, we mention [5, Corollary 2]: every metrically Ramsey ultrafilter on  $X$  has a member  $U$  with no subsets of the form  $\{x, y, x + y\}$ ,  $x \neq y$ .

In connection with Proposition 3.2, we ask

**Question 4.1.** *Let  $\mathcal{U}$  be a metrically Ramsey ultrafilter on  $\mathbb{N}$ . Does there exist a thin subset  $U \in \mathcal{U}$ ?*

**Question 4.2.** *Assume that a metrically Ramsey ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  has a thin member. Is  $\mathcal{U}$  selective?*

2. Let  $G$  be an Abelian group. A coloring  $\chi : [G]^2 \longrightarrow \{0, 1\}$  is called a *PS-coloring* if, for  $\{x, y\}, \{z, t\} \in [G]^2$ ,  $x + y = z + t$  implies  $\chi(\{x, y\}) = \chi(\{z, t\})$ . A free ultrafilter  $\mathcal{U}$  on  $G$  is called a *PS-ultrafilter* if  $\mathcal{U}$  is Ramsey with respect to all PS-colorings of  $[G]^2$ . The PS-ultrafilters were introduced and studied in [5], for exposition of [6] see [1, Chapter 10].

If  $G$  has a finite set  $B(G) = \{g \in G : 2g = 0\}$  of elements of order 2 then every PS-ultrafilter on  $G$  is selective. If  $B(G)$  is infinite then, under Martin's Axiom, there is a non-selective PS-ultrafilter on  $G$ . If there exists PS-ultrafilter on some countable group  $G$  then there is a  $P$ -point in  $\omega^*$ .

Now we consider the countable Boolean group  $\mathbb{B}$ ,  $B(\mathbb{B}) = \mathbb{B}$ . We note that a coloring  $\chi : [\mathbb{B}]^2 \longrightarrow \{0, 1\}$  is a PS-coloring if and only if  $\chi$  is  $\mathbb{B}$ -invariant. Thus, a free ultrafilter  $\mathcal{U}$  on  $\mathbb{B}$  is a PS-ultrafilter if and only if  $\mathcal{U}$  is  $\mathbb{B}$ -Ramsey. By above paragraph, in the models of ZFC with no  $P$ -points in  $\omega^*$ , there are no  $\mathbb{B}$ -Ramsey ultrafilters. However, every strongly summable ultrafilter on  $\mathbb{B}$  is a *PS-ultrafilter*. For strongly summable ultrafilters on Abelian groups see [3].

On the other hand,  $\mathbb{B}$  is the direct *sum*  $\bigoplus_{n < \omega} \{0, 1\}_n$  of  $\omega$  copies of  $\mathbb{Z}_2 = \{0, 1\}$ , and has the natural structure of ultrametric space

$(\mathbb{B}, d)$ , where  $d((x_n)_{n \in \omega}, (y_n)_{n \in \omega}) = \min\{m : x_n = y_n, n \geq m\}$ . By Proposition 2.1, there are plenty metrically Ramsey ultrafilters on  $\mathbb{B}$  in ZFC. Applying Proposition 1.6 (iii), we can find ultrafilters on  $\mathbb{B}$  which are not metrically Ramsey.

3. By [4, Theorem 6.2], there is a coloring  $\chi : [\mathbb{R}]^2 \longrightarrow \{0, 1\}$  such that if  $X \subset \mathbb{R}$  and  $[X]^2$  is  $\chi$ -monochrome then  $|X| \leq \omega$ .

We endow  $\mathbb{R}$  with the natural metric  $d(x, y) = |x - y|$  and ask

**Question 4.3.** *Does there exist an isometric coloring  $\chi : [\mathbb{R}]^2 \longrightarrow \{0, 1\}$  such that if  $[X]^2$  is monochrome then  $|X| \leq \omega$ ?*

We endow the Cantor cube  $\{0, 1\}^\omega$  with the standard metric and ask

**Question 4.4.** *Does there exist an isometric coloring  $\chi : [\{0, 1\}^\omega]^2 \rightarrow \{0, 1\}$  such that if  $[X]^2$  is monochrome then  $|X| \leq \omega$ ?*

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